Supplementary Remarks, by MR. FREDERICK HENDRIKS, "*On Auxiliary Tables for Life Contingencies," including Notice of a recent Table by* W. T. THOMSON, ESQ., F.R.S.E., *Manager of the Standard Life Assurance Company.*

HAVING been favoured by Mr. W. T. Thomson with a copy of his large Table, "Carlisle 3 per Cent.," and believing that the insertion of a notice of such laborious and useful computations as are brought together in that extensive work is in accordance with the objects of the *Assurance Magazine,* I beg to offer a brief explanation of some of the most important features of the Table: and such notice will afford an opportunity of referring to particulars which it appears desirable to annex to those submitted in my former paper.

In a communication from Mr. Thomson, that gentleman writes as follows:—

"Edinburgh, 19th October, 1850.

"Mr. Thomson observes, at page 18 of the *Assurance Magazine,* reference to the principle on which his Table is constructed, and requests particular attention to the fact that in his Table he actually deals with the *decrements* as well as with the living—for the difference between any two columns adjoining gives the value of the discounted decrements at the older age—and the 'conversion column' at the end of the Table is constructed on that principle, D column being discounted one year more to give the 'conversion column,' the difference being the C column. Mr. Thomson did begin a Table of discounted decrements, but soon found that the differences gave these, and abandoned it. For example (see large Table):—

Age 94	109·42538		
95	82·70815		
Difference = value of discounted decrements	26·71723	= ·890574	Value of reversion whole term.
(Living at 95)	30		

Again—

Age 94	80·29917		
95	60·37805		
Difference = value of discounted decrements deferred one year .	19·92112	= ·66404	Value of reversion deferred one year.
	30		

"Mr. Thomson thinks it right further to observe, that he was not aware of Teten's formulæ when he calculated his Table, and has had much pleasure in reading Mr. Hendrik's paper."

The annexed extract from the Table will serve as the example for present consideration and comparison:—

Ages.		103	102	101	100	99
No. living.		3	5	7	9	11
Ages.	No. Living.					
104	1	·97087	·94260	·91514	·88849	·86261
103	3	....	2·91262	2·82779	2·74542	2·66546
		Sum	3·85522	3·74293	3·63391	3·52807
102	5	....	....	4·85437	4·71298	4·57571
			Sum	8·59730	8·34689	8·10378
101	7	....	....	....	6·79612	6·59817
				Sum	15·14301	14·70195
100	9	....	....	....	....	8·73786
					Sum	23·43981
99	11	....	....	....	....	....

The thanks of those whom the matter concerns are due to Mr. Thomson for the industry and energy which have carried out the computation and printing of one of the most extensive collections of *details* ever brought forward with respect to a Table of Annuities on a *single life* at every age, and in that regard Mr. Thomson's Table completes the publications upon the Carlisle scale by Messrs. Milne, Sang, Gray, and others.

The formulæ which Mr. Thomson's Table represents may be thus described:

Let l_m = number of living in the Table at age m; l_{m+1} the number at age 1 year older than m; l_{m+2}, the number at 2 years older, and so on up to x years, or the difference between age m and the oldest life in the Table.

Let v = present value of £1 due at end of 1 year, v^2 the present value of £1 due at end of 2 years, and so on up to x years: then,

$$l_{m+1}v + l_{m+2}v^2 + \quad + l_{m+x}v^x$$

is equivalent to the aggregate present value at age m of as many separate Life Annuities as correspond *in number* with l_m, or the number of living at age m indicated by the Table; consequently as such aggregate present value is $= a_m . l^m$, it becomes the numerator of that fraction which is the elementary formula for computation of a single Life Annuity; viz.—

$$\frac{l_{m+1}v + l_{m+2}v^2 + \quad + l_{m+x}v^x}{l_m} = a_m,$$

and Mr. *Thomson's arrangement* (§3) is the detailed exposition of the component parts of the numerator of the above fraction, for every age of a single life, by the Carlisle observations and at 3 per cent. interest.

Dr. Halley's formula, expressed in the same notation, points out the identical arrangement; viz.—

$$\frac{l_{m+1}v}{l_m} + \frac{l_{m+2}v^2}{l_m} + + \frac{l_{m+x}v^x}{l_m} = a_m,$$

the denominator being *common* to all the numerators.

The following extract is from Dr. Halley's paper in the Philosophical Transactions for 1693, and to those who have not yet had an opportunity of its perusal, it will not be unprofitable, as it contains the first practical application of the theories of Probability and Interest to the computation of Life Annuities:—

"On this depends the valuation of Annuities on lives; for it is plain that the purchaser ought to pay for only such a part of the value of the Annuity, as he has chances that he is living; and this ought to be computed yearly, and the sum of all those yearly values being added together, will amount to the value of the Annuity for the life of the person proposed. Now the present value of money payable after a term of years, at any given rate of interest, either may be had from Tables already computed, or, almost as compendiously, by the Table of Logarithms, for the arithmetical complement of the logarithm of unity and its yearly interest being multiplied by the number of years proposed, gives the present value of one pound payable after the end of so many years. Then by the foregoing proposition it will be: as the number of persons living after that term of years, is to the number dead, so are the odds that any one person is alive or dead—and consequently as the sum of both, or the number of persons living of the age first proposed, is to the number remaining after so many years (both given by the Table), so is the present value of the yearly sum payable after the term proposed, to the sum which ought to be paid for the chance the person has to enjoy such an Annuity after so many years. And this being repeated for every year of the person's life, the sum of all the present values of these chances is the true value of the Annuity."

Dr. Halley thus professedly calculated each year's rent separately; but in so doing, as he used a *common denominator*, the question of one aggregate division or of separate partial divisions for subsequent summation, was so entirely dependent on the same principle, that the convenience or requirements of the computor were the only rules to be consulted in that respect, and where time was an object and the Annuity values required without those for sums deferred on attaining certain ages, the preferability of the aggregate final division was self-evident, and doubtless employed at that time by other computers of Annuity Tables; and the expediency of shortening such calculations did not escape at least the *wishes* of Dr. Halley, as may be seen in his Paper in the Philosophical Transactions, entitled, 'Some further Considerations on the Breslau Bills of Mortality (Anno 1693, No. 198).' The reader need scarcely be reminded that the formula above described was the only one available until the publication of Simpson's discovery of the method of deriving the value of a Life Annuity at a given age from the value at an age one year *older*. And that method was first published in 1742, in Simpson's 'Doctrine of Annuities and Reversions.' (See page 18 of 2nd edit. of 1775.)

Thus far only the latent existence of the arrangement has been referred to; we have next to consider a case where it was really employed.

In the year 1772 a Mr. William Dale published a work entitled 'Calculations deduced from first principles in the most familiar manner, by plain Arithmetic; for the use of the Societies instituted for the benefit of Old Age: intended as an Introduction to the Study of the Doctrine of Annuities. By a Member of one of the Societies.'

It would exceed a necessarily limited space to give any detailed account of the above book, which in some respects is singularly original, and it will be sufficient to direct attention to certain passages which, as it happens, are *most material* in the history of the origin of Auxiliary Tables for Life Contingencies.*

Dale's method of obtaining the values of temporary and deferred Life Annuities, which he takes nearly 40 pages to describe, is based entirely on the use of the partial summations of the numerators of the fraction

$$\frac{l_{m+1}v + l_{m+2}v^2 + \quad + l_{m+x}v^x}{l_m}$$

before alluded to (in the 6th section of the present paper) as representing Mr. Thomson's Table.

Dale's Annuity values are computed as payable half-yearly; this, of course, is a difference merely in the data applied.

The following abridged extract will give Dale's own view of the Table:—

"§ CVI. The method to find the value of an Annuity *during life* will be like that already used to find the value of an Annuity for a limited time, proceeding in the same manner to multiply the *number of survivors* into the *reversion* of the next half-year, and continuing it to the end of life; that is, as long as the Tables show that one is living. The total of all the sums added will show the capital stock that will pay the number of Annuitants *living* at the *commencement* of the Table, £2 Annuity during life.

"These calculations being made for the use of the Societies *in particular* who commence Annuitants at the age of fifty, it was not thought necessary to begin the Tables at a younger age.

* The way in which Dale treats his subject is both laborious and diffuse,—his admissions are ingenuous enough in that regard—and if those who are still found to advocate the explanation of mathematical rules without their generalization or reduction to formulæ and symbolic notation, would but read Dale's book *carefully through*, the process in many cases would lead to their entertaining a very different opinion. Let those, however, to whom the matter is new, first peruse the notice respecting a certain error of Dale, which is given in Baron Maseres' work, 'The Principles of the Doctrine of Life Annuities,' &c. London: 1783.

"*A Table to find the Value, in present money, of* £1 *immediate Annuity during life, from fifty to any age upwards, at the rate of* $3\frac{1}{2}$ *per cent. per annum, per half-yearly payments, and the Table of Mortality calculated from the London Bills for the last Forty-three years* (1728 *to* 1770).

Ages.	Living.	Reversion.	Product.	Totals.
50	221			
$50\frac{1}{2}$	$216\frac{1}{2}$	× ·9828009	= 212·77639485	4749·67697485
51	212	× ·965898	= 204·770376	4536·90058
*	*	*	*	*
60	140	× ·706824	= 98·95536	1850·7472315
$60\frac{1}{2}$	$136\frac{1}{2}$	× ·694668	= 94·822182	1751·7918715
*	*	*	*	*
$91\frac{1}{2}$	$2\frac{1}{2}$	× ·236943	= ·5923575	1·736835
92	2	× ·232865	= ·46573	1·1444775
$92\frac{1}{2}$	$1\frac{1}{2}$	× ·228862	= ·343293	·6787475
93	1	× ·224926	= ·224926	·3354545
$93\frac{1}{2}$	$\frac{1}{2}$	× ·221057	= ·1105285	·1105285

"The totals in the fifth or last column are added together in this manner:—The *product* at the age 93 is added to that opposite the age $93\frac{1}{2}$; the product opposite $92\frac{1}{2}$ is added to the total of the *two* last; the product opposite the age 92 is added to the total of the *three* last; the product of the age $91\frac{1}{2}$ is added to the total of the *four* last; and so on, in like manner, from the *end* to the *beginning;* where the total of *all* the payments is found to be 4749·67697485, which is the capital stock that will pay 221 persons, aged fifty, an immediate Annuity of £2 during life, according to *this* Table of Mortality.

"The use of this method is to show the sum that would pay the same number of persons, of the *same* age, the same Annuity, after any number of years' reversion. Suppose the age fifty *not* to commence Annuitant till the expiration of 10 years, it would receive none of the *first* 20 payments; and opposite the age $60\frac{1}{2}$, when he would become recipient of the *first* payment, is the sum of £1751·7918715, which is the capital stock that would pay 221 persons, aged fifty, an Annuity of £2 during life, to commence when aged 60. That stock divided by the 221 persons gives 7·92666 for £2, or £3·96333 for the value of £1 Annuity; and in like manner for any other time of reversion.

"By this Table, also, the value of an *immediate* Annuity for any age above *fifty* may be found thus:—What is the worth of an *immediate* Annuity of £1 for the age 60 during life?

"The total of all the payments *this* age would have a chance of receiving is opposite the age $60\frac{1}{2}$ in the fifth column, the same as before, 1751·7918715; but the sums are all multiplied into 20 half-years' reversion more than should be for an *immediate* Annuity for *this* age; therefore the total sum must be *divided* by 20 half-years' *reversion*, or *multiplied* by 20 half-years' *compound* interest, to restore it to *present* value; for the age is supposed by the question to receive the *first* payment at the expiration of 6 months, *not* at the expiration of 21 *half-years.*

·706824) 1751·7918715 (2478·398
or 1751·7918715 × 1·41477819 = 2478·398,

and let it be divided by the 140 persons supposed by this Table to be living at *sixty*, and the quotient will be 17·7028 for £2 Annuity, or £8·8514 for answer to the question."

The reader will understand that the preceding explanatory passages are applicable to Mr. Thomson's arrangement, which is identical with Dale's for a single life. Among the other uses of the Table is the obtaining of the values of Endowments on surviving a fixed term of years, by the formula $\frac{l_{m+n}.v^n}{l_m}$, also of the value of sums receivable at decease, by taking the difference between the summation at the given age and that at one year younger, and dividing by l_m the common denominator. In fact, the utility of the Table is so interwoven with the general *elementary* theory of Annuities and Assurances as not to require more than a reference to the latter to prove it.

Dale applied the method to *one* age only in each rate of mortality used, viz. (besides the Table above quoted), to age of 50 according to Dr. Halley's Breslau Observations (1687 to 1691), to age of 40 by Simpson's London Observations, published 1742, and to age of 50 by Dr. Price's London Table of 1759 to 1768.

Dale's Table thus contained the latent *type* of the D and N columns, which would have occurred *in reality* had he thought it *necessary* to compute but one of his Tables to be applicable to the *youngest age, i. e.* at 1 year *next birthday*, for in such case the number living at age $1 = l_1$ being multiplied into the present value of £1, receivable at end of one year $= v$, and l_2 by v_2, the product for any age m would have been $= l_m . v^m$ or Column D_m, and its summation

$$= l_m v^m + l_{m+1} v^{m+1} + + l_x v^x = N_m$$

(or N_{m-1} of more modern form); and *on such a principle* columns 4 and 5 of Dale's Table would have contained the complete foundation of a plan which was lost sight of by him in his *partial use* of his columns 2 and 5. Looking, however, to the character of his work, and to its candid representations, as to the disadvantages under which he laboured, it would seem almost certain that *had it so happened* that any necessary computation led to the D and N columns, it would not have induced his perception of having arrived at a general method. He doubtless possessed a large share of shrewdness and practical good sense; but, with similar qualities, the Schleswig Professor Tetens combined the advantages of cultivated and scientific attainments; and it would be difficult, indeed, to prove that such a man as Tetens required a *suggestive source* for a simple arrangement, which was an almost obvious result of the degree of study he devoted to generalising and elucidating the principles of the commutation method.

Before concluding, it may be desirable to annex an Example calculated by Tetens' Fifth Formula (see page 9 of this *Magazine*), and based on the Carlisle observations and rate of interest 3 per cent., for comparison with the extract at same ages by Mr. Thomson's arrangement, as given in page 13.

Ages.		103	102	101	100	99
No. living.		3	5	7	9	11
Ages.	Decrements or No. dying.					
104	1	·970874	1·913470	2·828611	3·717098	4·579707
103	2	....	1·941748	3·826940	5·657222	7·434196
102	2	....	....	1·941748	3·826940	5·657222
101	2	....	....	....	1·941748	3·826940
100	2	....	....	....	....	1·941748
99	2	....	....	....	....	....
	Sums	·970874	3·855218	8·597299	15·143008	23·439813

The final summations of the series given by the two methods are identical, and the Annuity values for the whole term of life are derived from the division of the sums by the number of living at the given age, as at the head of each column. The above Example will afford an excellent proof of the certainty of the manner in which the laws of probability are applied to the computation of Life Annuities. Let the above numerical results at age of 99 be considered. The Carlisle Table indicates (see 2nd column) that out of 11 persons living at that age, 2 certainly *die before* the end of one year, and a like number out of the original 11, *before* each of the 2nd, 3rd, 4th, and 5th years have respectively elapsed, leaving one survivor at age of 104 who does not outlive one year. An Annuity of £1 is consequently received by each for as many years certain as they survive *complete* years, the discounted value of an Annuity for 1 year certain has then to be multiplied by the number who die in the 2nd year, and the discounted values of Annuities *certain* for 2, 3, 4, and 5 years each, respectively, by the number who die in the 3rd, 4th, 5th, and 6th years. Such present discounted values being summed up, give the worth of 11 Annuities, payable to as many lives of the age of 99; equivalent, as above, to £23·439813; and one-eleventh of this sum, being £2·130892, is the present value of the single life Annuity at that age, according to the above record of observations.

8th Nov., 1850. F. H.

On the Contrivances required to render Contingent Reversionary Interests Marketable Securities.

WE have been favoured by Mr. Thomson, the Actuary of the Standard Life Assurance Company, and one of the Vice-Presidents of the Institute, with a copy of certain cases submitted to Mr. Edward Sang, and of his opinion in reference thereto. The matters to which they relate are of so much practical importance, and the learning and ability of Mr. Sang are so well known, that we gladly avail ourselves of Mr. Thomson's kind permission to publish